22 *I'm sorry, I don't understand this part of DATA line, so I'm ignoring it.* Note that Mx is not LISREL and won't fiddle about changing data structures using the MA= command syntax. You can always do it explicitly in a little Mx job which will be good for your immortal soul.

23 *Sorry, I couldn't invert your expected matrix.* If the method is ML, the expected matrix has to be invertible throughout optimization. Mx will try to avoid non-positive definite areas with a penalty function.

24 *Sorry, I couldn't invert your asymptotic matrix.* If the asymptotic matrix is not positive definite, it must be fixed. Check the Select command for repetitions of a number, if it is used.

25 *Strangely, you seem to have requested an unknown matrix operation.* IF you get this one, memory is screwed up somehow, and you should check your input file carefully before sending the problem to Mike Neale.

26 *Sorry, your model matrix has different dimensions from your data matrix.* A common silly mistake. Carefully figure out the dimension of your model, check NI parameter and Select command if used.

27 *Sorry, your expected matrix is not symmetric.* The matrix formula you provide should yield a matrix that is symmetric, if it is to be fitted to data.

28 ***Sorry, you must have symmetric or full model*** Same as previous message.

29 *Please try not to specify inverse for non-square matrices.* Generalized inverses are not available. If you are desperate, try transforming to a partitioned matrix that has a square matrix of full rank at one end.

30 *Sorry, had trouble transposing a matrix.* This is an unlikely error.

31 *Sorry, I couldn't find the determinant of a matrix.* I thought I put it on the shelf here somewhere... Probably zero.

32 *Uh-oh...* there's a problem with a binary operator. This can happen when evaluating an illegal matrix expression, but it is unlikely.

33 *Sorry, the matrix addition screwed up.* Very unlikely.

34 *The matrices you wish to add are not conformable for addition.* Quite likely. If at first you don't succeed, check check and check again

35 *The matrices you wish to multiply are not conformable for multiplication.* Be sure that you are using the right type of multiplication for your application, as well as checking the dimensions of the matrices you wish to multiply.

36 *You seem to have an unknown matrix type.* Very unlikely.

37 *I seem to be using an unknown fit function.* Very unlikely.

38 *You must specify 3 numbers for boundary constraints.* Lower limit, upper limit and parameter number - or matrix element.

39 *First character after BOUNDARY must be alphanumeric.* Alphanumeric means alphabetic abcdefghijklmnopqrstuv or 01234356789.

40 *Sorry, to request AWLS for this group, you should have input an asymptotic covariance matrix.*

41 *I don't know to what you want me to equate this matrix.* Some error in the =Mi command to equate matrices. Equatee must be on the same input line.

42 *Sorry, I can't make this matrix equal to a matrix that you haven't supplied yet.* Reorder your groups if it isn't a typo.

43 *Uh-oh... I got stuck inverting a matrix while calculating expected matrix...* This can be awkward. Sometimes starting values or changing the boundaries on parameters can help.

44 *Uh-oh... I'm having trouble reading a number in D or E format.* Probably a typo in the data.

45 *Sorry - could you put the =filename on the same line as the FI, please?*

46 *Awfully sorry, I couldn't open a file for you.* Probably a spelling mistake in the filename.

47 *I deeply regret that your equality constraint refers to a non-existent matrix.*

48 *OH NO! The IOP parameter calling MSOFAR is wrong.* I have no idea what this error means. The good thing about it is that you are not likely to get it.

49 *Please don't mix the numeric and array references on a Bound line!* Use only parameter numbers or only matrix elements.

50 *There I was, looking for a number and - blow me down...* Probably an error in specifying the dimensions of a matrix. The end of file was found before the number.

51 *It seems rather strange to me to have a + or - sign without a number after it.* Well, wouldn't it seem strange to you?

52 *You seemed to put a * in the middle of two numbers but WRONGLY.* Do not pass GO, do not collect $200

53 *Uh-oh. Now there are too many numbers in a list.*

54 *Nincompoop. Your data matrix has to be positive definite for GLS.* Probably a little harsh, this message.

55 *Ouch! do not try to change NG...* This used to cause big headaches.

56 *Awfully sorry, old chap, you're trying to | between matrices that...* Check conformability.

57 *Awfully sorry, old chap, you're trying to _ between matrices that...* Check conformability

58 *Uh-oh! Your formula has an illegal character.* Edit your input file and arrest this character immediately. On the IBM RISC 6000 it can occur spuriously and irrationally if you leave blank spaces at the start of a line following an underscore. Heaven knows why.

59 *Uh-oh! Your matrix expression has a mistake in it. Please fix it.* This could be unmatched parentheses, a missing operator or a missing matrix. Sorry that it isn't more specific... Also, the matrix formula is sensitive to untrapped memory problems. One known possibility is that you have tried to do something to a range of matrices from different groups, e.g. Start 1 A 1 1 1 - A 4 3 3

60 *There seems to a problem with your format in your data file.*

61 *Oh dear! The model you specified does not give the same number of rows...* Check the order of the model.

62 *Stack has overflowed - kick Mike Neale.* This is not necessary to fix the job. Your complicated expression ought to be simplified by using a CALC group to precalculate part of it.

63 *The matrices you wish to subtract are not conformable for subtraction.* If this message doesn't add up to you, go back to elementary school.

64 *I'm terribly sorry, stack 1 has overflowed.* Simplify your expression with CALC groups

65 *I'm terribly sorry, stack 2 has overflowed.* Simplify your expression with CALC groups.

66 *An undefined matrix has been encountered in the matrix formula.* Look for typos in the formula, and in the Matrices command. You're using a matrix that hasn't been specified for this group.

67 *You seem to have missed out an operator...* Check the matrix formula for matrix names that are more than one letter.

68 *At first you put % but then there was neither O nor E nor R after it.* So put O or E or R after it!

69 *You seem to be referring to parameter specifications for an Observed or Expected matrix...* There is no way that you are going to be allowed to do this.

70 *I want you to get this right.* If you equate to the Observed or... Get it right.

71 *The matrix you are trying to equate to the O matrix is bigger...* Make it smaller...

72 *The matrix you are trying to equate to the expected matrix is...* Make it the same size.

73 *You have to give NG after CA if the first group is a CALC group.* So do it.

74 *Tut-tut! you are trying to dot-product two matrices that have.* This is different from ordinary matrix multiplication after all.

75 *I tried to read another group and hit the end of file instead.* NG is probably wrong.

76 *Look buster, if you want to define your own function...* See Section 4.2.4

77 *At first I thought you were going to use a function, but then...* You must have boobed somewhere.

78 *Sorry, I only calculate the determinant of SQUARE matrices.*

79 *Sorry, I only calculate the trace of SQUARE matrices.*

80 *I say, if you use the \EXPonent operator, you must make sure that the...*You won't get this error message.

81 *I can't equate this matrix to the observed data of that group, because it...*

82 *Hmmm sorry I don't understand this keyword...*

83 *HEY! I thought you were going to say = but where's the =??*

84 *You can only have MA=CM,PM, or KM right now - sorry!*

85 *Just WHAT is this keyword supposed to mean?...* This is commonly encountered in the middle of a list of numbers if the list is too long. This may be because you have given too many numbers for the type and size of the matrix concerned, or alternatively you may have specified the dimensions of that matrix incorrectly. Note that you should only supply numbers for the modifiable elements of a matrix, which depends heavily on the type of matrix. See Section 3.3 for details on numbers of elements in different types of matrices.

86 *Imagine this: I'm reading a number and I see a character, so I think...* Do be less surprising in your input.

87 *I'm sorry, I can't write a matrix that doesn't exist to a file.* The matrix concerned has to be defined for that group.

88 *To output matrices to files, use* MX *with 1 letter after it (exceptions %E %M %P %V).*

89 *AAAAGH! You can't assign parameters to the raw data vector.*

90 *If you want to fit to the raw data vectors, you must put them in a vector.*

91 *No, I won't let you do this.* It would overwrite the first raw data observation...

92 *Uh-oh, attempt to take log of zero or negative value imminent.*

93 *In order to use Maximum Likelihood to raw data, it is necessary to supply both...*

94 *Sorry, the expected matrix is singular just now.*

95 *Just WHAT is this keyword supposed to mean?.*

96 *You have tried to convert something that wasn't a vector into a matrix.*

97 *At your starting values, evaluation of the log-likelihood made me take...* Nothing for it... change the starting values. This could be a problem if there are some gruesome outliers, in which case you'd have to edit your data... See Section 5.4.2 for details of how to interpret and respond to the diagnostics printed along with this error message.

98 *Your boundary constraint refers to a parameter not yet specified.*

99 *You have referred to a non-existent row of a matrix.* MX has a lousy imagination when it comes to that sort of thing.

100 *You have referred to a non-existent column of a matrix.* There are lots of non-existent columns in Athens and Rome.

101 *You can't use the keyword FU when reading diagonal weight matrices or means.*

102 *SORRY I don't select DWLS with correlation matrices yet.* Try again after midnight? Seriously, I'm sorry about this.

103 *Error - you seem to have a non-integer value for the type of person...* Make sure that you have got your VL file right. Be extra careful about SAS missing values.

104 *Please, you must tell me the covariance matrix structure in terms of id codes.*

105 *There is someone in this pedigree whose id has not been given.* The mysterious stranger needs to be identified and deleted.

106 *Your model does not define the COVARIANCE between TWO people with this ID.* See Section 2.3.3

107 *An element of a matrix that you have tried to \sqrt()...*

108 *You tried to convert a vector of the wrong length to a Symmetric matrix.* Hah, didn't think I'd notice eh?

109 *I am only prepared to raise the elements of a matrix to the power of a scalar.* Do you have any better ideas? Kroneckerize it or something like that?

110 *You tried to convert a vector of the wrong length to a FULL square matrix.* Silly billy.

111 *Eigenvalues of square matrices only, please*

112 *Eigenvectors of square matrices only, please*

113 *I'm afraid that if you want to fit to raw data you MUST supply models for BOTH means and covariances. You have forgotten a model for the means* Of course, you may have asked for the wrong kind of fit function on the OPTIONS line.

114 *I'm afraid that if you want to fit to raw data you MUST supply models for BOTH means and covariances. You have forgotten a model for the covariances* Of course, you may have asked for the wrong kind of fit function on the OPTIONS line.

115 *If you are using Multiple fits, it is impossible to change the matrix formulae - and you can't change boundaries either*

116 *If you wish to Specify, Pattern, or Matrix a matrix, you have to specify the group number **on the same line** BEFORE the Specification, Pattern or Matrix statement*

117 *Please try *not* to refer to non-existent groups!*

118 *When using the Multiple option you *MUST* use 3 numbers to specify a matrix element: Group #, row # and column #*

119 *I'm *so* embarassed. I ran out of workspace.*

120 *To read labels for a matrix, you must use the syntax: LABEL ¡R or C¿ Name+*

121 *Labels can't be given for a non-existent matrix* Actually they can, but then the program stops immediately.

122 *Labels may not begin with a number because it could confuse me later on.*

123 *Labels have not been provided for the data, but you seem to be using them to select variables. Use numbers or "Give me labels or give me ..."*

124 *It would seem that you are trying to select a variable that you never supplied. I expect that, being human, you made a mistake in the Labels or Select list*

125 *I can't select a variable that doesn't exist. Make sure that NI is correct or fix the Select list*

126 *Sorry, you can use covariance structures ONLY in constraint or calculation groups; means not allowed*

127 *Incorrect element of a lower triangular matrix specified*

128 *Hey! You must have NInputvars=2 to use contingency tables* A contingency table effectively cross-tabulates two variables, hence NInputvars must be two on the Data line.

129 *I'm afraid that if you want to fit to contingency tables you MUST supply models for BOTH thresholds and covariances. You have forgotten a model for the thresholds* Alternatively you may have requested the wrong fit function on the Options line.

130 *I'm afraid that if you want to fit to contingency tables you MUST supply models for BOTH thresholds and covariances. You have forgotten a model for the covariances* Perhaps you requested the wrong fit function on the Options line?

131 *You must supply a matrix expression for the thresholds that will evaluate to a matrix with 2 rows and with at least as many columns as one less than the number of row categories or the number of column categories, whichever is greater i.e. max(nrowcat-1,ncolcat-1)* See Section 3.2.4 for details about thresholds.

132 *Sorry, but during optimization I have been asked to calculate a bivariate integral with a correlation of 1 or more. This is very unreasonable of you Please fix your model so that this doesn't happen. Use boundaries or something.* Note that the *correlation* is critical here, not the covariance. The correlation is calculated from the expected covariance matrix (the result of the model or covariance statement) as $\mathrm{cov}_{ij}/\sqrt{\mathrm{var}_{ii}\mathrm{var}_{jj}}$

133 *Yes I will fit to n x n contingency tables BUT... n must be greater than or equal to 2* I know a $1 \times n$ is feasible but I haven't written the code for it yet, OK?

134 *If you want me to save or get an Mx binary file then you MUST supply a filename on the same line* It's just one of those things in life that you have to do.

135 *We must have misunderstood each other somehow. You can't get expected proportions %P+ from a group that isn't using contingency tables*

136 *I think memory is screwed up because Mike Neale has screwed up. Call him on 804 786 8590. Or better still, email him on* neale@ruby.vcu.edu *or* neale@vcuruby (bitnet). Has screwed up or is screwed up?

137 *Well well well. You would like graphics. If so why don't you put an = after the DR command???* Mx likes = signs before filenames. I don't know how it got into this habit.

138 *To fit to mean structures as well as covariances you should provide *both* observed means and a model for them (as well you know!) I'm going to use the covariances, but the mean - I'm just going to ignore it OK?* Watch out for this one if you don't seem to get any action with the means.

139 *Really I'm very very sorry about this. It is not under my control. Unless of course I was to use a better language than FORTRAN77 I suppose... However enough philosophising, the problem is that we have reached the end of file too soon. Since you are using * format, you should put one case per line. That is to say, there should be NObs lines each with NI variables in a RA data file* Sometimes error messages are not just explicit, they are introspective. How would you like to be a computer program? Could be our species' destiny.

140 *Really I'm very very sorry about this. It is not under my control. Unless of course I was to use a better language than FORTRAN77 I suppose... However enough philosophy, the problem is that we have reached the end of file too soon. Make sure that your format is OK' Also, remember there should be NObs * NI numbers in your RA data file* See previous error message for sci-fi remark.

141 *Very funny. Hahaha. You want me to standardize a non-square matrix? Just how am I supposed to do that?* If you get any ideas, let me know.

142 *It is with great sadness that I have to tell yo u that I couldn't standardize your matrix because the number I'm supposed to divide by is too small. Perhaps you could avoid this problem with boundary constraints.* Poor machine, it tried!

143 *You are in terrible danger. Don't say multiple until the last group. I know I could have just remembered for you, but I'm lazy too!* There are only so many hours in the day.

144 *Sorry, old chap. You can't specify boundary constraints after Options* I might remedy this problem one day, but for now just meekly go back and put boundary constraints before *the first* option line in this group.

145 *Cough, ahem... can you please give me a matrix that has 1 row and 2 column for the power transformation??* Currently it is only possible to apply a transformation to all variables within each vector. This should be upgraded if there is ever support for Mx; call your political representative now to safeguard its future...

146 *For heaven's sake! Can't you bound your constant so that it is *Greater* than minus the minimum observation??!* Really most nursery school children have a good idea why this should be done. Exponentiating (especially for non-integer exponents) numbers that are less than zero is mathematically awkward, requiring complex numbers and so on. To avoid complex arithmetic, whose implications are unclear to me in this context, Mx demands that if, say, your minimum observation is -3.1, the constant required would have to be greater than +3.1.

147 *YIKES! There's something funny about the power function you request. Remember that after PO you should supply 2 numbers. The significance level (alpha) (0.0 ¡ alpha ¡ 1), and the degrees of freedom of the test (df ¿ 0). Sometimes NAG chokes if alpha is close to 0 or 1* I can't think why this shouldn't be self-explanatory. The numbers should be on the same line as the PO keyword. Note that POwer in this context refers to calculating the statistical power of the study (see Section 5.2.2).

148 *I can calculate confidence intervals in the range 0 to 100. Please try to stay within these bounds'* Everyone has their limits, you know.

Some of these error messages are a little informal. I apologise. I'm terribly terribly sorry and I won't do it again.

What I am really sorry about is if Mx gives you the wrong error message. This is quite rare, but occasionally it is possible to screw up memory, and one of the sensitive areas is the matrix formula (since it contains junk instead of a reverse-polish version of your input formula). One untrapped source is if you say

```
start .5 A 1 2 3 to A 2 2 3
```

in which case everything between the memory addresses of A of group 1 and A of group 2 gets overwritten with .5 A little care can go a long way; so can a little more foolproof programing which I shall try to provide as soon as I can. Please let me know of any nonsensical error messages that you get; elucidation is prerequisite for elimination.

Appendix C

Introduction to Matrices

C.1 The Concept of a Matrix

A matrix is a table of numbers or symbols laid out in *rows* and *columns*.

$$\text{e.g.} \quad \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

The table is enclosed in () or [] in most texts.

It is conventional to specify the configuration of the matrix in terms of Rows × Columns and these are its *dimensions*. Thus the first matrix above is of dimensions 3 by 2 and the second is a 3 × 3 *square* matrix.

The most common occurrence of matrices in behavioral sciences is the *data matrix* where the *rows are subjects* and the *columns are measures.*

$$\text{e.g.} \quad \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cc} Wt. & Ht. \end{array} \\ \begin{bmatrix} 50 & 20 \\ 100 & 40 \\ 150 & 60 \\ 200 & 80 \end{bmatrix} \end{array}$$

It is convenient to let a single letter symbolize a matrix. This is written in UPPERCASE **boldface.**

Thus we might say that our data matrix is $\mathbf{A}$, which in handwriting we would underline with either a straight or a wavy line. Sometimes a matrix may be written ${}_4\mathbf{A}_2$ to specify its dimensions. When a matrix consists of a single number, it is called a *scalar*; when it consists of single column (row) of numbers it is called a column (row) *vector*. Vectors are normally represented as a **bold** lowercase. Thus the weights of our four subjects are

$$\begin{bmatrix} 50 \\ 100 \\ 150 \\ 200 \end{bmatrix} = \mathbf{a}$$

C.2 Matrix Algebra

Matrix algebra defines a set of operations that may be performed on matrices. These operations include addition, subtraction, multiplication, inversion (multiplication by the inverse is similar to division) and transposition.

C.2.1 Transposition

A matrix is transposed when the rows are written as columns and the columns are written as rows. This operation is denoted by writing $\mathbf{A}'$ or $\mathbf{A}^T$. In our example,

$$\mathbf{A}' = \begin{pmatrix} 50 & 100 & 150 & 200 \\ 20 & 40 & 60 & 80 \end{pmatrix}$$

a row vector is usually written

$$\mathbf{a}' = \begin{pmatrix} 50 & 100 & 150 & 200 \end{pmatrix}$$

Clearly, $(\mathbf{A}')' = \mathbf{A}$.

C.2.2 Matrix Addition and Subtraction

Matrices may be *added* and to do so they must be of the *same dimension.* They are then said to be *conformable for addition.* Each element in the first matrix is added to the corresponding element in the second matrix to form the same element in the solution.

$$\text{e.g.} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 8 & 11 \\ 9 & 12 \\ 10 & 13 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ 11 & 17 \\ 13 & 19 \end{pmatrix}$$

or symbolically,

$$\mathbf{A} + \mathbf{B} = \mathbf{C}.$$

You *cannot add*

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 8 & 10 \\ 9 & 11 \end{pmatrix}$$

Subtraction works in the same way as addition, e.g.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} - \begin{pmatrix} 2 & 5 \\ 2 & 5 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

which is written

$$\mathbf{A} - \mathbf{B} = \mathbf{C}.$$

C.2.3 Matrix Multiplication

Matrices *may be multiplied* and to do so they must be *conformable for multiplication.* This means that *adjacent columns and rows must be of the same order.* For example, the matrix product ${}_{\mathbf{3}}\mathbf{A}_{\mathbf{2}} \times {}_{\mathbf{2}}\mathbf{B}_{\mathbf{2}}$ may be calculated; the result is a 3×2 matrix. In general, if we multiply two matrices ${}_{\mathbf{i}}\mathbf{A}_{\mathbf{j}} \times {}_{\mathbf{j}}\mathbf{B}_{\mathbf{k}}$, the result will be of order $i \times k$.

Matrix multiplication involves calculating a *sum of cross products* among *rows of the first matrix* and *columns of the second matrix* in all possible combinations.

$$\text{e.g.} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1\times1+4\times2 & 1\times3+4\times4 \\ 2\times1+5\times2 & 2\times3+5\times4 \\ 3\times1+6\times2 & 3\times3+6\times4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 19 \\ 12 & 26 \\ 15 & 33 \end{pmatrix}$$

This is written

$$\mathbf{AB} = \mathbf{C}$$

The only exception to the above rule is multiplication by a *single number* called a scalar. Thus for example,

$$2\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ 4 & 10 \\ 6 & 12 \end{pmatrix}$$

by convention we write this

$$2\mathbf{A} = \mathbf{C}.$$

It is not possible to use this convention directly in Mx; however, it is possible to define a 1×1 matrix with the constant as the sole element, and use the kronecker product.

The simplest example of matrix multiplication is to multiply a vector by itself. If we premultiply a column vector by its transpose, the result is a scalar called the *inner product.* For example, if

$$\mathbf{a}' = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

then the inner product is

$$\mathbf{a}'\mathbf{a} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1^2 + 2^2 + 3^2 = 14$$

which is the sum of the squares of the elements of the vector $\mathbf{a}$. This has a simple graphical representation when $\mathbf{a}$ is of dimension 2×1 (see Figure 1).

C.2.4 Multiplication Exercises

Let

$$\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & -1 & -1 & 1 \end{pmatrix}$$

1. Form $\mathbf{AB}$.
2. Form $\mathbf{BA}$. (Careful, this might be a trick question!)

Let

$$\mathbf{C} = \begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

1. Form $\mathbf{CD}$.
2. Form $\mathbf{DC}$.
3. In ordinary algebra, multiplication is *commutative*, i.e. $xy = yx$. In general, is matrix multiplication commutative?
4. Show for two (preferably non-trivial) matrices conformable for multiplication that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

Let

$$\mathbf{E}' = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

1. Form $\mathbf{E}(\mathbf{C} + \mathbf{D})$.
2. Form $\mathbf{EC} + \mathbf{ED}$.
3. In ordinary algebra, multiplication is *distributive*, i.e. $x(y + z) = xy + xz$. In general, is matrix multiplication distributive?

C.3 Equations in Matrix Algebra

Matrix algebra provides a very convenient short hand for writing sets of equations. For example, the pair of *simultaneous equations*

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= x_1 + x_2 \end{aligned}$$

may be written

$$\mathbf{y}' = \mathbf{Ax}$$

i.e.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Also if we have the following pair of equations:

$$\mathbf{y} = \mathbf{Ax}$$

$$\mathbf{x} = \mathbf{Bz}$$

Then

$$\begin{aligned} \mathbf{y} &= \mathbf{A}(\mathbf{Bz}) \\ &= \mathbf{ABz} \\ &= \mathbf{Cz} \end{aligned}$$

where $\mathbf{C} = \mathbf{AB}$. This is very convenient notation compared with direct substitution. The LISREL structural equations are written in this general form, i.e.

Real variables (y) = Matrix $\times$ hypothetical variables.

To show the simplicity of the matrix notation, consider the following equations:

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= x_1 + x_2 \\ x_1 &= z_1 + z_2 \\ x_2 &= z_1 - z_2 \end{aligned}$$

Then we have

$$\begin{aligned} y_1 &= 2(z_1+z_2)+3(z_1-z_2) \\ &= 5z_1 - z_2 \\ y_2 &= (z_1+z_2)+(z_1-z_2) \\ &= 2z_1 + 0 \end{aligned}$$

From $\mathbf{y} = \mathbf{ABz}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$\mathbf{AB} = \begin{pmatrix} 5 & -1 \\ 2 & 0 \end{pmatrix},$$

or

$$\begin{aligned} y_1 &= 5z_1 - z_2 \\ y_2 &= 2z_2 \end{aligned}$$

C.4 Calculation of Covariance Matrix from Data Matrix

Suppose we have a data matrix $\mathbf{A}$ with rows corresponding to subjects and columns corresponding to variables. We can calculate a mean for each variable and replace the data matrix with a matrix of *deviations from the mean*. That is, each element a_{ij} is replaced by $a_{ij} - \mu_j$ where μ_j is the mean of the j^{th} variable. Let us call the new matrix $\mathbf{X}$. The covariance matrix is then simply calculated as:

$$\frac{1}{N-1}\mathbf{X}'\mathbf{X}$$

where N is the number of subjects.

For example, suppose we have the following data:

X	Y	$X-\bar{X}$	$Y-\bar{Y}$
1	2	-2	-4
2	8	-1	2
3	6	0	0
4	4	1	-2
5	10	2	4

So the matrix of deviations from the mean is

$$\mathbf{X} = \begin{pmatrix} -2 & -4 \\ -1 & 2 \\ 0 & 0 \\ 1 & -2 \\ 2 & 4 \end{pmatrix}$$

and therefore the covariance matrix of the observations is

$$\begin{aligned} \frac{1}{N-1}\mathbf{X}'\mathbf{X} &= \frac{1}{4}\begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ -4 & 2 & 0 & -2 & 4 \end{pmatrix}\begin{pmatrix} -2 & -4 \\ -1 & 2 \\ 0 & 0 \\ 1 & -2 \\ 2 & 4 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 10 & 12 \\ 12 & 40 \end{pmatrix} \\ &= \begin{pmatrix} 2.5 & 3.0 \\ 3.0 & 10.0 \end{pmatrix} = \begin{pmatrix} S_x^2 & S_{xy} \\ S_{xy} & S_y^2 \end{pmatrix} \end{aligned}$$

The correlation is

$$\frac{S_{xy}}{\sqrt{S_x^2 S_y^2}} = \frac{S_{xy}}{S_x S_y}$$

In general, a correlation matrix may be calculated from a covariance matrix by pre- and post-multiplying the covariance matrix by a *diagonal matrix* **D** in which each diagonal element d_{ii} is $\frac{1}{S_i}$, i.e. the reciprocal of the standard deviation for that variable. Thus in our two variable example, we have:

$$\begin{pmatrix} \frac{1}{S_x} & 0 \\ 0 & \frac{1}{S_y} \end{pmatrix}\begin{pmatrix} S_x^2 & S_{xy} \\ S_{xy} & S_y^2 \end{pmatrix}\begin{pmatrix} \frac{1}{S_x} & 0 \\ 0 & \frac{1}{S_y} \end{pmatrix} = \begin{pmatrix} 1 & R_{xy} \\ R_{xy} & 1 \end{pmatrix}$$

Transformations of Data Matrices

Matrix algebra provides a natural notation for *transformations*. If we premultiply the matrix ${}_{\mathbf{i}}\mathbf{B}_{\mathbf{j}}$ by another, say ${}_{\mathbf{k}}\mathbf{T}_{\mathbf{i}}$, then the rows of **T** describe linear combinations of the rows of **B**. The resulting matrix will therefore consist of k rows corresponding to the linear transformations of the rows of **B** described by the rows of **T**. A very simple example of this is premultiplication by the *identity matrix* (written **I**), which merely has 1's on the leading diagonal and zeroes everywhere else. Thus the transformation described by the first row may be written as 'multiply the first row by 1 and add zero times the other rows.' In the second row, we have 'multiply the second row by 1 and add zero times the other rows,' and so the identity matrix transforms the matrix $\mathbf{B}$ into the same matrix. For a less trivial example, let our data matrix be $\mathbf{X}$, then

$$\mathbf{X}' = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ -4 & 2 & 0 & -2 & 4 \end{pmatrix}$$

and let

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

then

$$\begin{aligned} \mathbf{Y}' &= \mathbf{T}\mathbf{X}' \\ &= \begin{pmatrix} -6 & 1 & 0 & -1 & 6 \\ 2 & -3 & 0 & 3 & -2 \end{pmatrix}. \end{aligned}$$

In this case, the transformation matrix specifies two transformations of the data: the first row defines the sum of the two variates, and the second row defines the difference (row 1 - row 2). In the above, we have applied the transformation to the raw data, but for these linear transformations it is easy to apply the transformation to the covariance matrix. The covariance matrix of the transformed variates is

$$\begin{aligned} \frac{1}{N-1}\mathbf{Y}'\mathbf{Y} &= \frac{1}{N-1}(\mathbf{T}\mathbf{X}')(\mathbf{T}\mathbf{X}')' \\ &= \frac{1}{N-1}\mathbf{T}\mathbf{X}'\mathbf{X}\mathbf{T}' \\ &= \mathbf{T}\mathbf{V}_{\mathbf{x}}\mathbf{T}' \end{aligned}$$

which is a useful result, meaning that linear transformations may be applied directly to the covariance matrix, instead of going to the trouble of transforming all the raw data and recalculating the covariance matrix.

Determinants

For a square matrix $\mathbf{A}$ we may calculate a scalar called the *determinant* which we write as $|\mathbf{A}|$. In the case of a 2×2 matrix, this quantity is calculated as

$$|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$$

The determinant has an interesting geometric representation. For example, consider two standardized variables that correlate r. This situation may be represented graphically by drawing two vectors, each of length 1.0, having the same origin and an angle cosine r between them (see figure 2). It can be shown (i.e. this is a toughie that involves symmetric square root decomposition of matrices, eigenvalues etc. that I'm not going to do here) that the area of the triangle OV_1V_2 is $.5\sqrt{|\mathbf{A}|}$. Thus as the correlation r increases, the angle between the lines decreases, the area decreases and *the determinant decreases.* For two variables that correlate perfectly, the determinant of the correlation (or covariance) matrix is zero. For larger numbers of variables, the determinant is a simple function of the hypervolume in n-space; if any single pair of variables correlates perfectly then the determinant is zero. In addition, if one of the variables is a linear combination of the others, the determinant will be zero.

To calculate the determinant of larger matrices, we employ the concept of a *cofactor.* If we delete row i and column j from an $n \times n$ matrix, then the determinant of the remaining matrix is called the *minor* of element a_{ij}. The cofactor, written A_{ij} is simply:

$$A_{ij} = (-1)^{i+j} \text{minor } a_{ij}$$

The determinant of the matrix $\mathbf{A}$ may be calculated as

$$|\mathbf{A}| = \sum_{i=1}^{n} a_{ij} A_{ij}$$

where n is the order of $\mathbf{A}$.

The determinant of a matrix is related to the concept of *definiteness* of a matrix. In general, for a null column vector $\mathbf{x}$, the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is always zero. For some matrices, this quadratic is zero *only* if $\mathbf{x}$ is the null vector. If $\mathbf{x}'\mathbf{A}\mathbf{x} > 0$ for all non-null vectors $\mathbf{x}$ then we say that the matrix is *positive definite.* Conversely, if $\mathbf{x}'\mathbf{A}\mathbf{x} < 0$ for all non-null $\mathbf{x}$, we say that the matrix is *negative definite.* However, if we can find some non-null $\mathbf{x}$ such that $\mathbf{x}'\mathbf{A}\mathbf{x} = 0$ then the matrix is said to be *singular*, and its determinant is zero. As long as no two variables are perfectly correlated, and there are more subjects than measures, a covariance matrix calculated from data on random variables will be *positive definite.* LISREL will complain (and rightly so!) if it is given a covariance matrix that is not positive definite. The determinant of the covariance matrix can be helpful when there are problems with model-fitting that seem to originate with the data. However, it is possible to have a matrix with a positive determinant yet which is negative definite (consider $-\mathbf{I}$ with an even number of rows), so the determinant is not an adequate diagnostic. Instead we note that all the eigenvalues of a positive definite matrix are greater than zero. Eigenvalues and eigenvectors may be obtained from software packages and the numerical libraries listed above[1].

[1]Those readers wishing to know more about the uses of eigenvalues and eigenvectors may consult Searle (1982) or any general text on matrix algebra.

Inverse of a Matrix

Just as there are many uses for the operation of division in ordinary algebra, there are many valuable applications of the inverse of a matrix. We write the inverse of the matrix $\mathbf{A}$ as $\mathbf{A^{-1}}$, and one of the most important results is that

$$\mathbf{A}\mathbf{A^{-1}} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. In this case, the multiplication operation is commutative, so it is also true that

$$\mathbf{A^{-1}}\mathbf{A} = \mathbf{I}$$

There are many computer programs available for inverting matrices. Some routines are general, but there are often faster routines available if the program is given some information about the matrix, for example whether it is symmetric, positive definite, triangular or diagonal. Here we describe one general method that everyone should use at least once in their lives for at least a 3×3 matrix.

Procedure for Inversion of a General Matrix

In order to invert a matrix, the following four steps can be used:

1. Find the determinant
2. Set up the matrix of cofactors
3. Transpose
4. Divide by the determinant

For example, the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$$

1.

$$|\mathbf{A}| = (1 \times 5) - (2 \times 1) = 3$$

2.

$$\begin{aligned} A_{ij} &= \begin{bmatrix} (-1)^2 \times 5 & (-1)^3 \times 1 \\ (-1)^3 \times 2 & (-1)^4 \times 1 \end{bmatrix} \\ &= \begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix} \end{aligned}$$

3.

$$A'_{ij} = \begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix}$$

4.

$$\mathbf{A^{-1}} = \frac{1}{3}\begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & \frac{-2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{pmatrix}$$

To verify this, we can multiply $\mathbf{AA^{-1}}$ to obtain the identity matrix:

$$\frac{1}{3}\begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The result that $\mathbf{AA^{-1}} = \mathbf{I}$ may be used to solve the pair of simultaneous equations:

$$\begin{aligned} x_1 + 2x_2 &= 8 \\ x_1 + 5x_2 &= 17 \end{aligned}$$

which may be written

$$\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$$

i.e.

$$\mathbf{Ax} = \mathbf{y}$$

premultiplying both sides by the inverse of $\mathbf{A}$, we have

$$\begin{aligned} \mathbf{A^{-1}Ax} &= \mathbf{A^{-1}y} \\ \mathbf{x} &= \mathbf{A^{-1}y} \\ &= \frac{1}{3}\begin{pmatrix} 5 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 17 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 6 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

which may be verified by substitution.

For a larger matrix it is more tedious to compute the inverse. Let us consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

1. The determinant is

$$\begin{aligned} |\mathbf{A}| &= +1\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ &= +1 + 1 + 0 \\ &= 2 \end{aligned}$$

2. The matrix of cofactors is:

$$\begin{aligned} A_{ij} &= \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} \\ &= \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 1 & -1 & -1 \end{pmatrix} \end{aligned}$$

3. The transpose is

$$A'_{ij} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

4. Dividing by the determinant, we have

$$\mathbf{A^{-1}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} .5 & 0 & .5 \\ .5 & 0 & -.5 \\ -.5 & 1 & -.5 \end{pmatrix}$$

which may be verified by multiplication with $\mathbf{A}$ to obtain the identity matrix.

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